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Projective and injective objects in the category of quantales ☆

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Abstract

Regular projective quantales are characterized as the weakly $*$-stable completely distributive lattices. For the class $\mathscr{E}$ of all onto quantale homomorphisms whose right adjoints preserve multiplication $*$, it is proved that $\mathscr{E}$-projective quantales are exactly weakly $*$-stable completely distributive lattices. It is also proved that there are no nontrivial injective objects in the category of quantales. © 2002 Published by Elsevier Science B.V.

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1. Introduction

Quantales were introduced by Mulvey in order to provide a lattice-theoretic setting for studying non-commutative C*-algebra, as well as constructive foundations for quantum mechanics (see Ref. [12,13]). The study of such partially ordered algebraic structures goes back to a series of papers by Ward and Dilworth (see Ref. [5,18,19]) in the 1930s. They were motivated by the ideal theory of commutative rings. Following Mulvey, various types and aspects of quantales have been considered by many researchers, see Ref. [16].

Even more recently, quantales have arisen in an analysis of the semantics of linear logic, logic systems developed by Girard [7], which supports part of the foundation

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of theoretic computer science, see Ref. [10,20]. Quantales also have links to the operational semantics and denotational semantics of computer languages, where they give the process semantics of varies kinds of observational equivalence, see Ref. [1,11]. So it is important to study the lattice properties of quantales. In this note, we study the projective and injective objects in the category of quantales.

Let $\mathscr{C}$ be a category and E a set of epimorphisms of $\mathscr{C}$, then an object c of $\mathscr{C}$ is said to be projective w.r.t E (simply E-projective) if for any morphism $f : c \to b$ and a morphism $g : a \to b$ in E, there is a morphism $\bar{f} : c \to a$ such that $f = g \circ \bar{f}$. Dually, we can define M-injective in a category $\mathscr{C}$ for a set M of monomorphisms of $\mathscr{C}$. Usually, we consider regular projectives and regular injectives in a category $\mathscr{C}$.

2. Regular projectives in the category of quantales

A quantale is a complete lattice Q with an associative binary multiplication $*$ satisfying

$$x * \left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I}(x * x_i) \quad \text{and} \quad \left(\bigvee_{i\in I} x_i\right) * x = \bigvee_{i\in I}(x_i * x)$$

for all $x, x_i \in Q$, $i \in I$, I is a set. 1 denotes the greatest element of Q, 0 is the smallest element of Q. A quantale Q is said to be unital if there is an element $u \in Q$ such that $u * a = a * u = a$ for all $a \in Q$.

Let P and Q be quantales, a function $f : P \to Q$ is a homomorphism of quantales if f preserves arbitrary sups and the operation $*$. If P and Q are unital, then f is a unital homomorphism if, in addition to being a homomorphism, it satisfies $f(u_P) = u_Q$, where u_P and u_Q are the respective units of P and Q.

Let **Quant** denote the category of quantales and homomorphisms and **UnQuant** be the category of unital quantales and unital homomorphisms. Then regular epimorphisms in **Quant** are exactly the onto homomorphisms, and monomorphisms in **Quant** are exactly the one-to-one (or embedding) homomorphisms. Thus, a quantale S is regular projective in **Quant** if for any homomorphism $f : S \to Q$ and an onto homomorphism $g : P \to Q$, there is a homomorphism $\bar{f} : S \to P$ such that $g \circ \bar{f} = f$. Dually, an injective in **Quant** can be given. For facts concerning quantales in general we refer to Ref. [16].

In any complete lattice L, let $a \lhd b$ (read as "a is wedge below b") means that, for any subset A of L, $b \leqslant \bigvee A$ implies that $a \leqslant t$ for some $t \in A$. Then it is well known that (see Ref. [6]) L is a completely distributive lattice (abbreviated by CDL in this paper) iff $x = \bigvee \{y | y \lhd x\}$ for all $x \in L$.

Definition 1. A quantale $(L, *)$ is called a $*$-stable CDL if it satisfies the following conditions:

(1) L is a CDL;
(2) $x_1 \lhd y_1, x_2 \lhd y_2 \Rightarrow x_1 * x_2 \lhd y_1 * y_2$;
(3) $x \lhd y_1 * y_2 \Rightarrow \exists x_1 \lhd y_1, \exists x_2 \lhd y_2$, such that $x = x_1 * x_2$.

A quantale $(L,*)$ is called a weakly $*$-stable CDL if it satisfies the above conditions (1), (2) and the following condition:
(4) $x \lhd y_1 * y_2 \Rightarrow \exists x_1 \lhd y_1, \exists x_2 \lhd y_2$, such that $x \leqslant x_1 * x_2$.

Obviously, a $*$-stable CDL is also a weakly $*$-stable CDL.

For any set X,the free quantale generated by X is just the powerset $\mathscr{P}(X^*)$ under subset inclusion, where X^* is the set of all finite nonempty strings of X, which is the free semigroup generated by X with the juxtaposition multiplication. The multiplication on $\mathscr{P}(X^*)$ is given by, $A * B = \{a * b | a \in A, b \in B\}$ for $A, B \in \mathscr{P}(X^*)$. Let $X_0^* = X^* \cup$ $\{$ the empty string $\}$, then X_0^* is the free monoid generated by X, and $\mathscr{P}(X_0^*)$ is the free unital quantale generated by X.

For any quantale $(S,*)$, the free quantale generated by the set S is $\mathscr{P}(S^*)$, with the map $\bigvee_S : \mathscr{P}(S^*) \to S$ defined by

$$\bigvee_S (A) = \bigvee_S \{x_1 * \cdots * x_n \in S | x_1 \ldots x_n \in S^*\}$$

then $\bigvee_S$ is obviously a quantale homomorphism. In the following, if there is no confusion, we do not distinguish between $x_1 \ldots x_n \in S^*$ and $x_1 * \cdots * x_n \in S$.

Proposition 1. *A quantale S is regular projective iff there exists a quantale homomorphism $h : S \to \mathscr{P}(S^*)$ such that $\bigvee_S \circ h = id_S$.*

Proof. Suppose that S is projective. The homomorphism $\bigvee_S$ is a regular epimorphism, so by the definition of projectives, for the identity map $id_S : S \to S$, there is a homomorphism $h : S \to \mathscr{P}(S^*)$ such that $\bigvee_S \circ h = id_S$.

Conversely, let $h : S \to \mathscr{P}(S^*)$ be a homomorphism satisfying $id_S = \bigvee_m \circ h$. Given any onto homomorphism $f : P \to Q$ and a homomorphism $g : \mathscr{P}(S^*) \to Q$. For each $s \in S$, there exists a $x_s \in P$ such that $g(\{s\}) = f(x_s)$. Then we obtain a map $k : S \to P$ by $k(s) = x_s$. Now $\mathscr{P}(S^*)$ is freely generated by the set S, there is a homomorphism $\bar{k} : \mathscr{P}(S^*) \to P$ such that $\bar{k} \circ \{\}_S = k$, where $\{\}_S : S \to \mathscr{P}(S^*)$ is the natural inclusion map. From S generates $\mathscr{P}(S^*)$ and $g \circ \{\}_S = f \circ k$, it follows that $g = f \circ \bar{k}$. This shows that $\mathscr{P}(S^*)$ is projective. Now let $f : P \to Q$ be any onto homomorphism and $g : S \to Q$ be any homomorphism. Then $g \circ \bigvee_S : \mathscr{P}(S^*) \to Q$ is a homomorphism from $\mathscr{P}(S^*)$ to Q, since $\mathscr{P}(S^*)$ is projective, there is a $j : \mathscr{P}(S^*) \to P$ such that $g \circ \bigvee_S = f \circ g$. Then the homomorphism $k = j \circ h : S \to P$ satisfies $f \circ k = (f \circ j) \circ h = g \circ \bigvee_S \circ h = g$. Hence, S is projective. □

Lemma 1. *If $h : L \to M$, $g : M \to L$ are quantale homomorphisms satisfying $g \circ h = id_L$, and M is a weakly $*$-stable CDL, then L is also a weakly $*$-stable CDL.*

Proof. First, we have the following implication:

$$y \lhd h(x) \text{ in } M \Rightarrow g(y) \lhd x \text{ in } L. \; (*)$$

This is because, if $x \leqslant \bigvee_L A$ in L , then $h(x) \leqslant h\left(\bigvee_L A\right) = \bigvee_M h(A)$ in M. Since $y \lhd h(x)$ in M, $y \leqslant h(a)$ for some $a \in A$, and then $g(y) \leqslant gh(a) = a$ for some $a \in A$, hence $g(y) \lhd x$ in L. Since M is a CDL, $h(b) = \bigvee\{y \in M | y \lhd h(b)\}$ for all $b \in L$, and then $b = gh(b) = \bigvee\{g(y) \in L | y \lhd h(b)\}$. From $(*)$ it follows that $b = \bigvee\{l \in L | l \lhd b\}$, thus L is a CDL.

Furthermore, if $a_1 \lhd b_1, a_2 \lhd b_2$ in L, from $b_i = \bigvee\{g(y_i) | y_i \lhd h(b_i)\}$, it follows that $a_i \leqslant g(y_i)$ for some $y_i \lhd h(b_i)$, then $a_1 * a_2 \leqslant g(y_1) * g(y_2) = g(y_1 * y_2)$, and $y_1 * y_2 \lhd h(b_1) * h(b_2) = h(b_1 * b_2)$. By the hypotheses of g, M and h, the latter inequality implies $g(y_1 * y_2) \lhd b_1 * b_2$ in L, and hence we have $a_1 * a_2 \lhd b_1 * b_2$.

Finally, if $a \lhd b_1 * b_2$ in L, since $b_1 * b_2 = \bigvee\{g(y) | y \lhd h(b_1 * b_2)\}$, there is $y \lhd h(b_1) * h(b_2)$ such that $a \leqslant g(y)$. Since M is a $*$-stable CDL and $y \lhd h(b_1) * h(b_2)$ in M, there are $y_1 \lhd h(b_1)$ and $y_2 \lhd h(b_2)$ such that $y \leqslant y_1 * y_2$, then from $(*)$, it follows that $a_1 = g(y_1) \lhd b_1, a_2 = g(y_2) \lhd b_2$ and $a \leqslant g(y) \leqslant g(y_1) * g(y_2) = a_1 * a_2$.

Hence, L is also a weakly $*$-stable CDL. □

Lemma 2. *The free quantale $\mathscr{P}(X^*)$ generated by X is a $*$-stable CDL.*

Proof. (1) $\mathscr{P}(X^*)$ is a power set of X^*, and hence a CDL.

(2) $x \lhd A$ in $\mathscr{P}(X^*)$ iff $x \in A$ for some $x \in X^*$, hence $x_1 \lhd A_1, x_2 \lhd A_2 \Rightarrow x_1 \in A_1, x_2 \in A_2 \Rightarrow x_1 * x_2 \in A_1 * A_2 \Rightarrow x_1 * x_2 \lhd A_1 * A_2$.

(3) If $x \lhd A_1 * A_2$ then $x \in A_1 * A_2 = \{x_1 * x_2 | x_1 \in A_1, x_2 \in A_2\}$, and thus $x = x_1 * x_2$ for some $x_1 \in A_1, x_2 \in A_2$, that is, $x = x_1 * x_2$ for some $x_1 \lhd A_1, x_2 \lhd A_2$.

Therefore, $\mathscr{P}(X^*)$ is a $*$-stable CDL. □

Theorem 1. *If S is a regular projective quantale, then S is a weakly $*$-stable CDL; and if S is a $*$-stable CDL, then S is a regular projective quantale.*

Proof. If S is projective, then from Lemmas 1 and 2, S is a weakly $*$-stable CDL.

If S a $*$-stable CDL, by Proposition 1 if we can prove that $\bigvee_S : \mathscr{P}(S^*) \to S$ has a left inverse $h : S \to \mathscr{P}(S^*)$, i.e. $\bigvee_S \circ h = id_S$, then S is regular projective quantale.

Define $h : S \to \mathscr{P}(S^*)$ as follows:

$h(s) = \{x \in S^* | x \lhd s \text{ in } S\}$.

We check that h is a quantale homomorphism in the following.

Since $h(\bigvee s_i) = \{x \in S^* | x \lhd \bigvee s_i \text{ in } S\} = \{x \in S^* | x \lhd s_i \text{ for some } i \text{ in } S\} = \bigcup\{x \in S^* | x \lhd s_i \text{ in } S\} = \bigcup h(s_i)$, it follows that $h(\bigvee s_i) = \bigvee h(s_i)$.

Since $h(x_1 * s_2) = \{x \in S^* | x \lhd s_1 * s_2 \text{ in } S\} = \{x \in S^* | \exists x_1 \lhd s_1, x_2 \lhd s_2, \text{ such that } x = x_1 * x_2 \text{ in } S\}$, and $h(s_1) * h(s_2) = \{x_1 * x_2 \in S^* | x_1 \lhd s_1, x_2 \lhd s_2\}$, it follows that $h(s_1 * s_2) = h(s_1) * h(s_2)$.

Furthermore, since S is a CDL, $\bigvee_S \circ h(s) = \bigvee_S\{x \in S | x \lhd s\} = s$ for any $s \in S$, i.e., $\bigvee_S \circ h = id_S$. □

We further give some characterization of weakly $*$-stable CDL in the following.

Let A be a partially ordered set with an associate binary multiplication $*$ satisfying

$$x \leqslant y \Rightarrow x * z \leqslant y * z \quad \text{and} \quad z * x \leqslant z * y$$

for all $x, y, z \in A$. Then A is called a partially ordered semigroup (abbreviated by po-semigroup in this paper). For a po-semigroup A, let DA denote the down-set lattice of partially ordered set A, that is, the lattice of all $X \subseteq A$ such that $a \in X$ implies $b \in X$ for all $b \leqslant a$. DA is closed under arbitrary unions and intersections, hence a complete lattice, and XY in DA iff $X \subseteq \downarrow a$ for some $a \in Y$. Multiplication on DA is defined as follows:

$$X * Y = \downarrow \{x * y | x \in X, y \in Y\} = \{z \in A | \exists x \in X, \exists y \in Y \text{ such that } z \leqslant x * y\}$$

for all $X, Y \in DA$. Then, it is an easy task to check that under the above multiplication DA becomes a quantale. Concerning the quantale, we have the following proposition to characterize weakly $*$-stable CDL.

Lemma 3. *For any po-semigroup A, DA defined as above is a weakly $*$-stable CDL.*

Proof. First, since DA is a sub-complete lattice of powerset lattice $\mathscr{P}(A)$, DA is a CDL.

Second, if $X_1 \lhd Y_1$ and $X_2 \lhd Y_2$ in DA, then $X_1 \subseteq \downarrow a_1$ and $X_2 \subseteq \downarrow a_2$ for some $a_1 \in Y_1$ and $a_2 \in Y_2$,it follows that $X_1 * X_2 \subseteq \downarrow a_1 * \downarrow a_2 = \downarrow (a_1 * a_2)$ for $a_1 * a_2 \in Y_1 * Y_2$, and thus $X_1 * X_2 Y_1 * Y_2$.

Third, if $X \lhd Y_1 * Y_2$, then $X \subseteq \downarrow (a_1 * a_2) = \downarrow a_1 * \downarrow a_2$ for some $a_1 \in Y_1, a_2 \in Y_2$. Let $X_1 = \downarrow a_1, X_2 = \downarrow a_2$, then $X \leqslant X_1 * X_2$ and $X_1 \subseteq \downarrow a_1, X_2 \subseteq \downarrow a_2$ for some $a_1 \in Y_1, a_2 \in Y_2$, the latter inclusion relation implies that $X_1 \lhd Y_1$ and $X_2 \lhd Y_2$.

Hence, DA is a weakly $*$-stable CDL. $\square$

If S is a quantale, then S is also a po-semigroup. In this case, we have an onto map $\bigvee_S : DS \to S$ defined as follows:

$$\bigvee_S(X) = \bigvee_S X$$

for any $X \in DS$. It is readily verified that $\bigvee_S$ is a homomorphism.

Proposition 2. *A quantale S is a weakly $*$-stable CDL iff there is a right inverse of $\bigvee_S$, that is, there is a homomorphism $h : S \to DS$ such that $\bigvee_S \circ h = id_S$.*

The proof is similar to that of Theorem 1, we omit it here.

Furthermore, weakly $*$-stable CDLs are just certain projective objects in the category of quantales which we describe in the following.

For two po-semigroups A and B, a function $f : A \to B$ is called a po-semigroup homomorphism if f preserves multiplication and order. Let **Pg** denote the category of all po-semigroups with po-semigroups homomorphisms. Then obviously, **Quant** is a (non-full) subcategory of **Pg**, and the corresponding $A \to DA$ defines a functor D : **Pg** $\to$ **Quant** such that, for any $f : A \to B$ in **Pg**, $Df : DA \to DB$ take each $X \in DA$ to $\bigcup \downarrow f(a)(a \in X)$, that is, the downset in B generated by $f(X)$. Concerning the functor, and let $\mathscr{E}$ denote all the onto quantale homomorphisms whose right adjoints preserve multiplication, we have the following two lemmas.

Lemma 4. *D is left adjoint to the inclusion functor* **Quant** $\rightarrow$ **Pg**.

Proof. It suffices to show that $\downarrow: A \rightarrow DA$ is universal among the po-semigroup homomorphisms from A into quantales for a quantale A. Assuming that $f: A \rightarrow B$ is a po-semigroup homomorphism for a quantale B, define $\bar{f}: DA \rightarrow B$ as $\bar{f}(D) = \bigvee_B \{f(d) | d \in D\}$ for any $D \in DA$. Then it is readily verified that $\bar{f}$ is a unique quantale homomorphism satisfying $\bar{f} \circ \downarrow = f$. $\square$

Lemma 5. *Each $DA, A \in$* **Pg**, *is $\mathscr{E}$-projective in the category of quantales.*

Proof. Consider the diagram

$$\begin{tikzcd} L \arrow[r, "f"] & M \\ A \arrow[u, "h"] \arrow[r, "\downarrow"] & DA \arrow[u, "g"'] \end{tikzcd}$$

where $h = f^{\vee} \circ g \circ \downarrow, f^{\vee}$ the right adjoint of f, given by $f^{\vee}(z) = \bigvee\{x | f(x) \leqslant z\}$. Let $k: DA \rightarrow L$ be the unique quantale homomorphism such that $k \circ \downarrow = h$, resulting from Lemma 4 and the fact that h is a po-semigroup homomorphism because $f^{\vee}$ is. Now, $f \circ f^{\vee} = id_M$ whenever f is surjective, and then

$$f \circ k \circ \downarrow = f \circ h = f \circ f^{\vee} \circ g \circ \downarrow = g \circ \downarrow,$$

which shows $g = f \circ k$. Hence DA is $\mathscr{E}$-projective. $\square$

Combining the above lemma and Proposition 2, we can obtain the following:

Theorem 2. *The $\mathscr{E}$-projective objects in* **Quant** *are exactly the weakly $*$-stable CDLs.*

Remark 1. Using the same technique, we can give the same results for the regular projective objects and $\mathscr{E}$-projective objects in the category **UnQuant**. We only add extra the condition that $u \lhd u$ for the unit of quantale in the definition of (weakly) $*$-stable CDL.

Problem. *Is there a weakly $*$-stable CDL which is not a $*$-stable CDL?*

Examples. (1) Any complete lattice L can be made into a quantale by taking $x * y = y$ for all $x, y \in L$ with $y \neq 0$ and $x * 0 = 0$. Then $*$-quantales are exactly complete lattices with sups-preserving mappings, the later category is denoted by *Sup*. In this case,

$$\text{weakly } *\text{-stable CDLs} = *\text{-stable CDLs} = \text{CDLs},$$

hence, the regular projective objects in **Sup** are CDLs, which was shown in [4].

(2) If the meet operation distributes to multiplication, that is, if the equation

$$a \wedge (b * c) = (a \wedge b) * (a \wedge c),$$

holds, then it is readily verified that the notion of a weakly $*$-stable CDL is the same as that of a $*$-stable CDL. In these cases, the regular projective quantales are exactly weakly $*$-stable CDLs. In particular, if $* = \wedge$, finite meets, then $*$-quantales are just frames with frame homomorphisms (for general theory of frame, see Ref. [9]). In this case, the weakly $*$-stable CDLs are exactly stably supercontinuous lattices defined by Banaschewski in Ref. [2,3], and they are exactly the regular projective objects in the category of frames.

(3) The unit interval $[0,1]$ with the usual order is a complete chain and thus a CDL. If $*$ is an associative and commutative binary multiplication which is also non-decreasing in its first variable, that is, $x \leqslant y \Rightarrow x * z \leqslant y * z$, then $*$ is called a generalized triangular norm, which is an extension of t-norm (with unit 1) and t-conorm (with unit 0) (see Ref. [17]). If a generalized triangular norm $*$ is a continuous (w.r.t. the usual topology of $[0,1]$), e.g., product or minimum operation, then $([0,1],*)$ is a quantale.

The product operation $\cdot$ is a continuous t-norm on $[0,1]$: we check that $([0,1],\cdot)$ is a $\cdot$-stable CDL as follows. Note that $x \lhd y$ in $[0,1]$ iff $x < y$ in $[0,1]$. If $x_1 < y_1$ and $x_2 < y_2$ in $[0,1]$, then $x_1x_2 < x_1y_2 < y_1y_2$. And if $x < y_1y_2$, choose $x_1 < y_1$ such that $x < x_1y_2$ and let $x_2 = x/x_1$, then $x_2 < y_2$ and $x = x_1x_2$. Therefore, $([0,1],\cdot)$ is a $\cdot$-stable CDL.

In fact, if $*$ is any continuous t-norm (or t-conorm) on $[0,1]$ such that the mapping $a * x : [0,1] \to [0,1]$ is strictly increasing for any fixed $a \in (0,1)$, then $([0,1],*)$ is a $*$-stable CDL. Since $1 \lhd 1$ is not true in $[0,1]$, these examples are not regular projective in the category of unital quantales.

3. Supercoherent quantales and their coreflections in the category of quantales

Definition 2. Let $(Q,*)$ be a quantale, for $a \in Q$, if $a \lhd a$, then a is called a supercompact element of Q. Let SQ denote all the supercompact elements of Q. If Q satisfies the following conditions:

(1) for all $a \in Q$, $a = sup\{c \in SQ | c \leqslant a\}$;
(2) if $c_1, c_2 \in SQ$, then $c_1 * c_2 \in SQ$,
then Q is called a supercoherent quantale.

Obviously, supercoherent quantale must be weakly $*$-stable CDL.

Let **SCohQuant** denote the category of supercoherent quantales with those quantale homomorphisms that preserve supercompact elements. Let **WSCDL** denote the category of weakly $*$-stable CDLs with those quantale homomorphisms that preserve the relation $\lhd$. We note that, for supercoherent quantales S and Q, $h : S \to Q$ preserves $\lhd$ iff it preserves supercompact elements since, in this case, $a \lhd b$ iff $a \leqslant c \leqslant b$ for some supercompact element c.

From Lemma 4, the functor D : **Pg** $\to$ **Quant** actually induces an equivalence between **Pg** and **SCohQuant**. It is clear that D actually goes into that category, and to get back consider the functor S : **ScohQuant** $\to$ **Pg** for which SQ is the po-semigroup of supercompact elements of Q. Once then readily checks that $\downarrow: A \to S(DA)$ and $\bigvee : D(SQ) \to Q$ are natural isomorphisms.

Now, we have the following:

Proposition 3. **SCohQuant** *and* **WSCDL** *are both coreflective in* **Quant**, *with the coreflection functor D and the coreflection map $\bigvee : DQ \to Q$.*

Proof. Since all DQ are supercoherent and **WSCDL** contains **SCohQuant**, it is sufficient to show that any quantale homomorphism $h : S \to Q$ with weakly $*$-stable CDL S uniquely factors through $\bigvee : DQ \to Q$ by a $\lhd$-preserving quantale homomorphism. For this, consider the diagram

$$\begin{tikzcd} DQ \arrow[r, "\vee_Q"] & Q \\ DS \arrow[u, "Dh"] \arrow[r, shift left, "\vee_S"] & S \arrow[u, "h"'] \arrow[l, shift left] \end{tikzcd}$$

where k is given by $k(a) = \{x \in S | x \lhd a\}$. As noted earlier (Theorem 1), k is a quantale homomorphism, and hence we have the quantale homomorphism $\bar{h} = Dh \circ k : S \to DQ$ such that $\bigvee_Q \circ \bar{h} = \bigvee_Q \circ Dh \circ k = h \circ \bigvee_S \circ k = h \circ id_S = h$ since $\bigvee_S \circ k = id_S$. Moreover, $a \lhd b$ in S implies $\downarrow a \subseteq k(b)$, hence $\bar{h}(a) \subseteq \downarrow h(a) \subseteq \bar{h}(b)$, and therefore $\bar{h}(a) \lhd \bar{h}(b)$. To see uniqueness, take any $f : S \to DQ$, of the type in question, such that $\bigvee \circ f = h$. Then $x \lhd a$ in S implies $f(x) \lhd f(a)$ in DQ, and hence $f(x) \subseteq \downarrow c \subseteq f(a)$ for some $c \in Q$. It follows that $h(x) \leqslant c$ by taking joins, and this shows $\bar{h}(a) \subseteq f(a)$. On the other hand, for any $z \in f(a)$, $\downarrow z \lhd f(a)$ and hence $\downarrow z \subseteq f(x)$ for some $x \lhd a$ since S is a CDL. But then $z \leqslant h(x)$, hence $z \in \bar{h}(a)$, proving that also $f(a) \subseteq \bar{h}(a)$. In all, we therefore have that $f = \bar{h}$. $\square$

Corollary 1. *The weakly $*$-stable CDLs are exactly the retracts of supercoherent quantales.*

Remark 2. The above notions and results can be generalized to unital quantales, and then we can obtain similar coreflection results.

4. Injective objects in the category of quantales

Lemma 6. *Every quantale can be embedded into a unital quantale.*

Proof. Given a quantale S, define $\bar{S} = S \times \{0, 1\}$ as a complete lattice with the cartesian product of S and $\{0, 1\}$, where $0, 1 \in S$. Define the multiplication on $\bar{S}$ as follows,

$$(r_1, k_1) * (r_2, k_2) = (r_1 * r_2 \vee (r_1 \wedge k_2) \vee (r_2 \wedge k_1), k_1 \wedge k_2).$$

It is an easy task to check that $\bar{S}$ is a unital quantale with the unit $(0, 1)$. The embedding $m : S \to \bar{S}$ is defined as follows:

$$m(r) = (r, 0)$$

for all $r \in S$. $\square$

Note that the above construction corresponds to the extension of a non-unital ring to a unital ring (see Ref. [8]). In fact, the above construction $\bar{S}$ is quantale isomorphic to the construction $K[e]$ given in Ref. [14].

Lemma 7. *If $e : Q \to S$ is an onto quantale homomorphism and Q has an unit u, then S has unit an $e(u)$.*

The proof is easy.

Corollary 2. *Every injective quantale is unital.*

Proof. Combining Lemmas 6 and 7 deduce the result. □

For any complete lattice L, let $Sup(L)$ denote the unital quantale of sup-preserving map from L to itself, with sups of mappings computed pointwise, with multiplication given by composition of mappings, and with the unit given by the identity mapping. Then $Sup(L)$ is a simple quantale, as studied in Ref. [14,15], that is, any surjective homomorphism of quantale from it is either an isomorphism or a constant homomorphism.

Theorem 3. *There are no nontrivial injective quantales.*

Proof. Suppose S is a injective quantale, then S is unital. Note that the map $m : S \to Sup(S)$ defined as follows:

$$m(a)(b) = a * b$$

is a quantale embedding (for details, see Ref. [14]), from the definition of injective quantale, it follows that there is a onto homomorphism $e : Sup(S) \to S$ such that $e \circ m = id_S$. Since $Sup(S)$ is a simple quantale, e is an isomorphism or a constant homomorphism. If e is a constant homomorphism, then S is a singleton quantale. If e is an isomorphism, then m is also an isomorphism. Note m is not an onto map if S contains more than one element, hence S is a singleton quantale. Therefore, the injective quantale must be a singleton. □

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